JOURNAL OF ALGEBRA **210,** 86–102 (1998)
ARTICLE NO. JA987582

Flat Covers of Complexes

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Communicated by Craig Huneke

Received April 3, 1997

In this article we define and study flat complexes over any ring. Also, we prove that any complex over a commutative noetherian ring with finite Krull dimension has a flat cover and a DG-flat cover. © 1998 Academic Press

1. PRELIMINARIES

In this article $\mathscr{C}$ will be the abelian category of complexes of left R-modules. This category has enough projectives and injectives. This can be seen from the fact that any complex of the form

$$\cdots \to 0 \to M \xrightarrow{\text{id}} M \to 0 \to \cdots,$$

with M projective (injective) is projective (injective). This, in turn, follows from the fact that a complex (C, δ) is projective (respectively, injective) in $\mathscr{C}$ if and only if it is exact, and the module $\operatorname{Ker} \delta^i$ is projective (respectively, injective), for all $i \in \mathbb{Z}$ (see [2]). For objects C and D of $\mathscr{C}$, $\operatorname{Hom}_{\mathscr{C}}(C, D)$ is the abelian group of morphisms from C to D in $\mathscr{C}$ and $\operatorname{Ext}^i_{\mathscr{C}}(C, D)$ for $i \geq 0$ will denote the groups we get from the right-derived functors of $\operatorname{Hom}_{\mathscr{C}}$.

*Supported by DGICYT Grant PB95-1065.

0021-8693/98 $25.00

In this paper a complex

$$\cdots \to C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots$$

will be denoted C. We will use subscripts to distinguish complexes. So if $\{C_i\}_{i \in I}$ is a family of complexes, C_i will be

$$\cdots \to C_i^{-1} \xrightarrow{\delta^{-1}} C_i^0 \xrightarrow{\delta^0} C_i^1 \xrightarrow{\delta^1} \cdots .$$

If C and D are complexes we let $\mathscr{H}\mathrm{om}(C, D)$ be the complex of abelian groups with

$$\mathscr{H}\mathrm{om}(C, D)^n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}(C^t, D^{n+T})$$

and such that if $f \in \mathscr{H}\mathrm{om}(C, D)^n$ then

$$(\delta^n f)^m = \delta_D^{n+m} \circ f^m - (-1)^n f^{m+1} \circ \delta_C^m .$$

Then $Z^0 \mathscr{H}\mathrm{om}(C, D)$ will be the group $\mathrm{Hom}_{\mathscr{C}}(C, D)$ of morphisms from C to D.

Given M a left R-module, we will denote by $\overline{M}$ the complex

$$\cdots 0 \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to 0 \cdots ,$$

with the M on the right-hand side in the zeroth position. Also we mean by $\underline{M}$ the complex with M in the zeroth place and 0 in the other places. Given a complex C and integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m \delta^{m+n}$.

We note that $\underline{R}$ is a subcomplex of the projective complex $\overline{R}$ with quotient $\underline{R}[1]$. An element of $\mathrm{Hom}_{\mathscr{C}}(\underline{R}, C)$ corresponds to an element $x \in Z^0(C)$ which will be a boundary of C if and only if the corresponding morphism $\underline{R} \to C$ can be extended to $\overline{R} \to C$. Hence $\mathrm{Ext}^1_{\mathscr{C}}(\overline{R}[1], C) \cong H^0(C)$. More generally, $\mathrm{Ext}^1_{\mathscr{C}}(\overline{R}[n], C) \cong H^{-n+1}(C)$.

2. FLAT COMPLEXES

DEFINITION 2.1. (1) We will say that a complex C is finitely generated if, in case $C = \sum_{i \in I} D_i$, with $D_i \in \mathscr{C}$ subcomplexes of C, then there exists a finite subset $J \subseteq I$ such that $C = \sum_{i \in J} D_i$.

(2) We will say that a complex C is finitely presented if C is finitely generated and for any exact sequence of complexes $0 \to K \to L \to C \to 0$ with L finitely generated, K is also finitely generated.

LEMMA 2.2. (i) *C is finitely generated if and only if C is bounded and C^n is finitely generated in R-Mod for all $n \in \mathbb{Z}$.*

(ii) *C is finitely presented if and only if C is bounded and C^n is finitely presented in R-Mod for all $n \in \mathbb{Z}$.*

Proof. (i) Suppose that C is finitely generated in $\mathscr{C}$. If C is not bounded we can find the following infinite family of bounded subcomplexes of C:

$$\{0 \to C^0 \to B^0(C) \to 0, 0 \to C^{-1} \to C^0 \to C^1 \to B^1(C) \to 0, \dots\}$$

whose sum is C and if C is a finite sum of these complexes we get that C is bounded, a contradiction.

Suppose now that C^k is not finitely generated in R-Mod. Then $C^k = \sum_{i \in I} D_i$ with D_i nonnull submodules of C^k and C^k cannot be written as a finite sum of some terms of the family $\{D_i\}_{i \in I}$. We consider the following family of submodules of C^k: $\{D_J\}_{J \in \mathscr{F}}$, where $\mathscr{F} = \{J \subseteq I: |J| < \infty\}$ and $D_J = \sum_{j \in J} D_j$. Then $\sum_{J \in \mathscr{F}} D_J = \bigcup_{J \in \mathscr{F}} D_J = C^k$. We construct the family of subcomplexes of C, $\{\overline{D}_J\}_{J \in \mathscr{F}}$, where

$$\overline{D}_J \equiv \cdots C^{k-2} \to (\delta^{k-1})^{-1}(D_J) \to D_J \to C^{k+1} \to \cdots .$$

Then it is clear that the sum of this family is C and we cannot find a finite subfamily in $\{D_J\}_{J \in \mathscr{F}}$ whose sum is C, a contradiction.

Suppose now that C is bounded and C^i is finitely generated for all $i \in \mathbb{Z}$. Let $\{D_j\}_{j \in I}$ be a family of subcomplexes of C such that its sum is C. Then for each i we consider $C^i = \sum_{j \in J_i} D_j^i$ with $J_i \subseteq I$ and $|J_i| < \infty$. We take $K = \bigcup J_i$. Since C is bounded the set K is finite and it is clear that $\sum_{i \in K} D_i = C$. So C is a finitely generated complex.

(ii) Suppose first that C is finitely presented in $\mathscr{C}$. By [10, Theorem 2.4], we can find an epimorphism $F \to C$ with F bounded and F^m free for all $m \in \mathbb{Z}$. Then the induced short exact sequence $0 \to K \to F \to C \to 0$ has K finitely generated in $\mathscr{C}$. Hence for any $m \in \mathbb{Z}$ we have a short exact sequence

$$0 \to K^m \to F^m \to C^m \to 0,$$

with F^m free and finitely generated and K^m finitely generated. So C^m is finitely presented for all $m \in \mathbb{Z}$.

Conversely, suppose that C^m is finitely presented and C is bounded. It is clear that C is finitely generated in $\mathscr{C}$. So let $0 \to K \to L \to C \to 0$ be a short exact sequence in $\mathscr{C}$ with L finitely generated. Then K is bounded because it is a subcomplex of a bounded complex. Also, for any $m \in \mathbb{Z}$,

$$0 \to K^m \to L^m \to C^m \to 0$$

has C^m finitely presented and L^m finitely generated. Therefore K^m is finitely generated for all $m \in \mathbb{Z}$. ∎

PROPOSITION 2.3. *Let* $(F, \omega) \equiv \cdots F^{-2} \to F^{-1} \to F^0 \to 0 \cdots$ *be an exact complex, bounded above and with* $K^i = \operatorname{Ker}(\omega^i)$ *flat in R-Mod for all* $i \in \mathbb{Z}$. *Let* $f\colon C \to F$ *be a morphism with* (C, δ) *a finitely generated complex. Then there is a finitely generated projective complex P and a factorization of* f, $C \xrightarrow{\alpha} P \xrightarrow{\beta} F$.

Proof. For $i > 0$ we will take $P^i = 0$, $\alpha^i = 0$, and $\beta^i = 0$. By [9, Theorem 1.2], we have a factorization of f^0, $C^0 \to^{\alpha^0} P^0 \to^{\beta^0} F^0$, with P^0 finitely generated and projective. Since $\omega^{-1}\colon F^{-1} \to F^0$ is an epimorphism we find $h^0\colon P^0 \to F^{-1}$ such that $\omega^{-1}h^0 = \beta^0$. Then $\omega^{-1}h^0\delta^{-1} = f^0\delta^{-1} = \omega^{-1}f^{-1}$. So $t^{-1} = f^{-1} - h^0\alpha^0\delta^{-1}$ is a map from C^{-1} to K^{-1}. Since K^{-1} is flat and C^{-1} finitely generated, there is a factorization of t^{-1}:

$$C^{-1} \xrightarrow{j^{-1}} Q^{-1} \xrightarrow{d^{-1}} K^{-1},$$

with Q^{-1} finitely generated and projective. Therefore we can construct the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q^{-1} & \longrightarrow & Q^{-1} \oplus P^0 & \longrightarrow & P^0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^{-1}} & & \downarrow{\scriptstyle d^{-1}+h^0} & & \downarrow{\scriptstyle \beta^0} & & \\ 0 & \longrightarrow & K^{-1} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & 0 \end{array}$$

and we get the factorization of f^{-1}:

$$C^{-1} \xrightarrow{j^{-1}+\alpha^0\delta^{-1}} Q^{-1} \oplus P^0 \xrightarrow{d^{-1}+h^0} F^{-1}.$$

We say $Q^{-1} \oplus P^0 = P^{-1}$, $j^{-1} + \alpha^0\delta^{-1} = \alpha^{-1}$, and $d^{-1} + h^0 = \beta^{-1}$.

By induction, suppose we have constructed α^j, β^j, P^j for $j > i$ and now we construct α^i, β^i, P^i. Since $\omega^i\colon F^i \to K^i$ is an epimorphism we get a map $h^{i+1}\colon Q^{i+1} \to K^i$ such that $\omega^i h^{i+1} = d^{i+1}$. Then $\omega^i f^i = \omega^i h^{i+1} j^{i+1}\delta^i$. So we have a factorization of $t^i = f^i - h^{i+1}j^{i+1}\delta^i\colon C^i \to K^i$, $C^i \to^{j^i} Q^i \to^{d^i} K^i$ where Q^i is finitely generated and projective. Now we consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q^i & \longrightarrow & Q^i \oplus Q^{i+1} & \longrightarrow & Q^{i+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^i} & & \downarrow{\scriptstyle d^i+h^{i+1}} & & \downarrow{\scriptstyle d^{i+1}} & & \\ 0 & \longrightarrow & K^i & \longrightarrow & F^i & \longrightarrow & K^{i+1} & \longrightarrow & 0 \end{array}$$

and take $P^i = Q^i \oplus Q^{i+1}$, $\alpha^i = j^i + j^{i+1}\delta^i$, $\beta^i = d^i + h^{i+1}$. In this way, we can construct a factorization with the desired properties. ∎

THEOREM 2.4. *Let* (F, δ) *be a complex of left* R*-modules. We denote* $K^i = \operatorname{Ker}(\delta^i)$ *for all* $i \in \mathbb{Z}$. *The following conditions are equivalent.*

(1) *F is exact and K^i is flat for all $i \in \mathbb{Z}$.*

(2) *For each finitely generated complex C and any morphism $C \to F$ there is a factorization $C \to P \to F$ where P is a finitely generated projective complex.*

(3) *F is a direct limit of finitely generated projective complexes.*

(4) *F^+ is an injective complex of right R-modules where*

$$F^+ \equiv \cdots \to (F^i)^+ \to (F^{i-1})^+ \to \cdots$$

and $(-)^+ = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$.

(5) *For any short exact sequence in* $\mathscr{C}$,

$$0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} F \to 0$$

and any morphism $f\colon D \to F$ with D finitely presented, there exists $g\colon D \to M$ such that $\beta g = f$.

Proof. (1) $\Rightarrow$ (2) This is Proposition 2.3 with the obvious modifications.

(1) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) If F^+ splits then F is exact and also $(K^i)^+$ are injective implies that K^i are flat.

(3) $\Rightarrow$ (1) Suppose $F = \varinjlim P_i$ where P_i are finitely generated and projective complexes. Since P_i are exact, it follows that F is exact. Also $Z^k(F) = Z^k(\varinjlim P_i) = \varinjlim Z^k(P_i)$ and $Z^k(P_i)$ are projective, so the $\operatorname{Ker}(\delta^k)$ are flat for all $k \in \mathbb{Z}$.

(2) $\Rightarrow$ (3) First note that F is a direct union of bounded above complexes

$$\{\cdots F^{i-1} \to F^i \to K^i \to 0 \cdots\}_{i \in \mathbb{Z}},$$

which satisfy the same conditions that F does. So we can suppose that F is bounded above, $F \equiv \cdots F^{-2} \to F^{-1} \to F^0 \to 0 \cdots$.

Consider a set of finitely generated projective complexes and morphisms, $\{(P, f)\}$ with $f\colon P \to F$, where the set includes every such pair (P, f) up to isomorphism. Let $D = \oplus(P, f)$ (with the sum over all pairs (P, f)) and let $g\colon D \to F$ be given by the f's in the pair (P, f). Let $\overline{D} = \oplus_{j=0}^{\infty} D_j$ with $D_j = D$ and let $\overline{D} \to F$ agree with g on each D_j.

Consider the directed set (S, U) where U is the sum of a finite number of summands (P, f) (for various (P, f) and various j), and where $S \subset U$ is a finitely generated subcomplex of U with $S \subset \operatorname{Ker}(\overline{D} \to F)$. We order the pairs by $(S, U) \leq (S', U')$ if and only if $S \subset S'$ and $U \subset U'$. Then clearly $\varinjlim U/S \cong F$.

We need a cofinal subset of pairs (S, U) such that U/S is projective. Given (S, U), U/S is finitely generated in $\mathscr{C}$ so by hypothesis there is a factorization $U/S \to F$ as

$$U/S \xrightarrow{\bar{h}} \overline{C} \xrightarrow{\bar{f}} F,$$

with $\overline{C}$ finitely generated and projective, where we can suppose $(\overline{C}, \bar{f})$ is in our original set of pairs. Each of the summands C in U is a summand of some D_i, so let $n_0 \neq i$ for all such i.

Let $\overline{U}$ be the sum of U and let $\overline{C}$ be a summand of D_{n_0} (i.e., it is the $(\overline{C}, \bar{f})$ component). We write $\overline{U} = U \oplus \overline{C}$ (understood in the obvious way). Let h be the morphism

$$U \to U/S \xrightarrow{\bar{h}} \overline{C}$$

and let $\overline{S}$ be the subcomplex of $\overline{U}$ given by $\overline{S}^n = \{(u^n, -h^h(u^n)) \mid u^n \in U^n\}$ for all $n \in \mathbb{Z}$. Then $\overline{U}/\overline{S} \cong \overline{C}$ and $\overline{S} \subseteq \operatorname{Ker}(\overline{D} \to F)$. If $u^n \in S^n$ then $h^n(u^n) = 0$ so $S \subseteq \overline{S}$. Hence $(S, U) \leq (\overline{U}, \overline{S})$. (We note that this proof closely follows Lazard [9].)

(1) $\Rightarrow$ (5) Let $0 \to N \to^{\alpha} M \to^{\beta} F \to 0$ be a short exact sequence in $\mathscr{C}$. Take D a finitely presented complex and $f\colon D \to F$ a morphism. We denote (M, γ), (F, δ), and

$$(D, \omega) \equiv \cdots 0 \to D^0 \to \cdots \to D^t \to 0 \cdots .$$

We are going to construct a morphism $g\colon D \to M$ such that $\beta g = f$. We define, for $j > t$, $g^j = 0$. For $j = t$ we have $\delta^t f^t = 0$. Hence $f^t\colon D^t \to K^t$. We get $h^t\colon D^t \to F^{t-1}$ such that $\delta^{t-1} h^t = f^t$ (K^{t-1} is pure and D^t is finitely presented). Since F^{t-1} is flat we find $v^t\colon D^t \to M^{t-1}$ such that $\beta^{t-1} v^t = h^t$. Define $g^t\colon D^t \to M^t$ as $g^t = \gamma^{t-1} v^t$. Then $\beta^t g^t = \beta^t \gamma^{t-1} v^t = \delta^{t-1} \beta^{t-1} v^t = \delta^{t-1} h^t = f^t$. Also $\gamma^t g^t = \gamma^t \gamma^{t-1} v^t = 0$.

For $j = t-1$ we have $\delta^{t-1} f^{t-1} = f^t \omega^{t-1}$ and $\delta^{t-1} h^t = f^t$. Hence $\delta^{t-1}(f^{t-1} - h^t \omega^{t-1}) = 0$. Denote $d^{t-1} = f^{t-1} - h^t \omega^{t-1}\colon D^{t-1} \to K^{t-1}$. We get a map $h^{t-1}\colon D^{t-1} \to F^{t-2}$ such that $\delta^{t-2} h^{t-1} = d^{t-1}$. Also we find $v^{t-1}\colon D^{t-1} \to M^{t-2}$ such that $\beta^{t-2} v^{t-1} = h^{t-1}$. We define $g^{t-1}\colon D^{t-1} \to M^{t-1}$ as $g^{t-1} = \gamma^{t-2} v^{t-1} + v^t \omega^{t-1}$. Then $\beta^{t-1} g^{t-1} = \beta^{t-1} \gamma^{t-2} v^{t-1} + \beta^{t-1} \; v^t \omega^{t-1} = \delta^{t-2} \; \beta^{t-2} v^{t-1} + h^t \; \omega^{t-1} = \delta^{t-2} h^{t-1} + h^t \omega^{t-1} = f^{t-1} - h^t \omega^{t-1} + h^t \omega^{t-1} = f^{t-1}$. Hence $\beta^{t-1} g^{t-1} = f^{t-1}$. Also, $\gamma^{t-1} g^{t-1} = \gamma^{t-1} \gamma^{t-2} v^{t-1} + \gamma^{t-1} v^t \omega^{t-1} = g^t \omega^{t-1}$.

Suppose we have constructed g^l for $i < l$ $(i < t-1)$ and we are going to construct g^i. From the equalities $\delta^i f^i = f^{i+1}\omega^i$ and $\delta^i h^{i+1} + h^{i+2}\omega^{i+1} = f^{i+1}$ we have $\delta^i(f^i - h^{i+1}\omega^i) = 0$; denote $d^i = f^i - h^{i+1}\omega^i$: $D^i \to K^i$. Now the rest of the proof follows in the same way as in the case $j = t-1$.

(5) $\Rightarrow$ (1) We know that there exists an exact sequence

$$0 \to K \xrightarrow{\alpha} P \xrightarrow{\beta} F \to 0$$

in $\mathscr{C}$ with P projective such that

$$0 \to \operatorname{Hom}_{\mathscr{C}}(D, K) \to \operatorname{Hom}_{\mathscr{C}}(D, P) \to \operatorname{Hom}_{\mathscr{C}}(D, F) \to 0$$

is exact for all finitely presented complexes D. If we prove that K is exact, it will follow that F is exact.

The complex $\underline{R}[n]$ is finitely presented, so the sequence

$$0 \to \operatorname{Hom}_{\mathscr{C}}(\underline{R}[n], K) \to \operatorname{Hom}_{\mathscr{C}}(\underline{R}[n], P) \to \operatorname{Hom}_{\mathscr{C}}(\underline{R}[n], F) \to 0$$

is exact. Then we have a monomorphism

$$0 \to \operatorname{Ext}^1_{\mathscr{C}}(\underline{R}[n], K) \to \operatorname{Ext}^1_{\mathscr{C}}(\underline{R}[n], P).$$

But P is an exact complex. Hence $\operatorname{Ext}^1_{\mathscr{C}}(\underline{R}[n], P) \cong H^{-n+1}(P) = 0$, so

$$H^{-n+1}(K) \cong \operatorname{Ext}^1_{\mathscr{C}}(\underline{R}[n], K) = 0$$

for all $n \in \mathbb{Z}$. Therefore K is an exact complex.

Now we have to prove that the K^i are flat for all $i \in \mathbb{Z}$. First we show that the F^i are flat for all $i \in \mathbb{Z}$. Let f: $H \to F^n$ with H finitely presented in R-Mod. We consider the complex $\overline{H}[-n]$. Then we take the morphism $\bar{f}$: $\overline{H}[-n] \to F$ with $\bar{f}^n = f$, $\bar{f}^{n+1} = \delta^n f$, and $\bar{f}^k = 0$ for all $k \neq n, n+1$. Since $\overline{H}[-n]$ is a finitely presented complex, there exists g: $\overline{H}[-n] \to P$ such that $\beta g = \bar{f}$. Hence $\beta^n g^n = f$. Therefore the sequence $0 \to K^n \to P^n \to F^n \to 0$ is pure. Since P^n is projective, it follows that F^n is flat.

Now we show that the sequences $0 \to K^i \to F^i \to K^{i+1} \to 0$ are pure in R-Mod. Let f: $D \to K^{n+1}$ be a map with D finitely presented in R-Mod. We consider the complex $\underline{D}[-n-1]$ and the morphism $\bar{f}$: $\underline{D}[-n-1] \to F$ with $\bar{f}^{n+1} = f$ and $\bar{f}^i = 0$ for all $i \neq n+1$. Then there exists g: $\underline{D}[-n-1] \to P$ such that $\beta g = \bar{f}$. Then $\beta^{n+1} g^{n+1} = f$ and $\omega^{n+1} g^{n+1} = 0$ where

$$P \equiv \cdots P^i \xrightarrow{\omega^i} P^{i+1} \xrightarrow{\omega^{i+1}} P^{i+2} \cdots .$$

We have the map g^{n+1}: $D \to \mathrm{Ker}(\omega^{n+1}) = \mathrm{Im}(\omega^n)$. Hence there exists υ^n: $D \to P^n$ such that $\omega^n \upsilon^n = g^{n+1}$ (note that w^n: $P^n \to \mathrm{Im}(\omega^n)$ splits). Hence $\delta^n \beta^n \upsilon^n = \beta^{n+1} \omega^n \upsilon^n = \beta^{n+1} g^{n+1} = f$ and so the sequence

$$0 \to K^n \to F^n \to K^{n+1} \to 0$$

is pure in R-Mod. Since F^n is flat, it follows that K^{n+1} is flat for all $n \in \mathbb{Z}$. ▮

Definition 2.5. Let (F, δ) be a complex. We will say that F is flat if it verifies one of the equivalent conditions of Theorem 2.4.

Remark. In a flat complex F each module F^i is flat, but the converse is not true in general. However, the converse does hold if F is bounded above and exact.

3. FLAT COVERS OF COMPLEXES

Let $\mathscr{A}$ be a class of objects in an abelian category $\mathscr{C}$. We recall the definition introduced in [4].

Definition 3.1. Let X be an object of $\mathscr{C}$. An $\mathscr{A}$-precover of X is a morphism ϕ: $E \to X$ such that E is in $\mathscr{A}$, and every triangle

$$\begin{tikzcd} E' \arrow[d, dotted] \arrow[dr, "\phi'"] & \\ E \arrow[r, "\phi"'] & X \end{tikzcd}$$

with E' in $\mathscr{A}$ can be completed. We say that ϕ is an $\mathscr{A}$-cover if a triangle with $\phi' = \phi$ can be completed only by automorphisms. An $\mathscr{A}$-precover ϕ: $E \to X$ is said to be special (or a special precover) if $\mathrm{Ext}^1(E', \mathrm{Ker}(\phi)) = 0$ for all $E' \in \mathscr{A}$.

Lemma 3.2. *Let ϕ: $F \to C$ be a flat precover in $\mathscr{C}$. Then ϕ^n: $F^n \to C^n$ is a flat precover in R-Mod for all $n \in \mathbb{Z}$.*

Proof. Let D be a flat left R-module and let f: $D \to C^n$ be a linear map. We consider the induced morphism $\bar{f}$: $\overline{D}[-n] \to C$. Since $\overline{D}[-n]$ is a flat complex there is a morphism h: $\overline{D}[-n] \to F$ such that $\phi h = \bar{f}$. So $\phi^n h^n = f$. ▮

Remark. (1) A flat cover of a complex must be an epimorphism.

(2) Let $(C, \delta) \equiv \cdots 0 \to C^0 \to C^1 \to \cdots$ be a bounded below complex and let ϕ: $(F, \gamma) \to (C, \delta)$ be a flat cover. It is easy to see that $\overline{\phi}$:

$\overline{F} \to C$ is a flat precover where

$$\left(\overline{F}, \gamma\right) \equiv \ \cdots 0 \to \operatorname{Ker}\left(\gamma^0\right) \to F^0 \to F^1 \ \cdots \ .$$

Since F is a direct summand of $\overline{F}$ it follows that $F^i = 0$ for $i < -1$ and $\operatorname{Ker}(\gamma^0) = F^{-1}$. Suppose that C^0 is not flat in R-Mod. Since $\phi^0\gamma^{-1} = 0$ it follows that $\operatorname{Ker}(\gamma^0) \subseteq \operatorname{Ker}(\phi^0)$ and $F^0/\operatorname{Ker}(\gamma^0)$ is flat. So ϕ^0: $F^0 \to M^0$ is not a flat cover if $\operatorname{Ker}(\gamma^0) \neq 0$.

EXAMPLES. (1) If ϕ: $F \to M$ is a flat cover in R-Mod, then $\overline{\phi}[n]$: $\overline{F}[n] \to \overline{M}[n]$ is a flat cover in $\mathscr{C}$ for all $n \in \mathbb{Z}$.

(2) Let R be a right coherent ring. Suppose that the left R-module M has a flat cover ϕ: $F \to M$. Then ϕ: $[F] \to \underline{M}$ is a flat cover, where

$$[F] \equiv 0 \to F \to PE^0(F) \to PE^1(F) \to \ \cdots$$

is a pure-injective resolution of F. Note that, in this case, all the $PE^n(F)$'s are flat.

(3) If ϕ: $F \to M$ is a flat cover in R-Mod, then the induced morphism

$$\phi'\colon \overline{F} \to \left(\cdots 0 \to F \xrightarrow{\phi} M \to 0 \cdots \right)$$

is a flat cover in $\mathscr{C}$.

Recall that a left R-module N is said to be *cotorsion* if $\operatorname{Ext}^1(F, N) = 0$ for every flat left R-module F.

DEFINITION 3.3. (i) We say that a complex (C, δ) is DG-cotorsion if $\operatorname{Ext}^1_{\mathscr{C}}(F, C) = 0$ for any flat complex F.

(ii) We say that (C, δ) is cotorsion if C is exact and $\operatorname{Ker}(\delta^i)$ is cotorsion in R-Mod for all $i \in \mathbb{Z}$.

Remark. (1) From example (2) it is clear that a DG-cotorsion complex does not have to be exact and does not have to have $\operatorname{Ker}(\delta^i)$ cotorsion in R-Mod (by Wakamatsu's lemma [12, Prop. 2.22] the kernel of a flat cover is always DG-cotorsion). Also, it can be shown that if C is cotorsion then C is DG-cotorsion.

(2) If $Y \in R$-Mod is cotorsion and $L \to C$, $F \to M$ are flat covers in R-Mod, then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & L \oplus F & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is a flat cover in $\mathscr{C}$ for all short exact sequences $0 \to Y \to X \to M \to 0$. Also the kernel of this cover is cotorsion.

PROPOSITION 3.4. (1) *(C, δ) is DG-cotorsion if and only if C^n is cotorsion in R-Mod and $\mathscr{H}\mathrm{om}(F, C)$ is exact for any flat complex F.*

(2) *If (C, δ) is bounded below then (C, δ) is DG-cotorsion if and only if C^n is cotorsion for all $n \in \mathbb{Z}$.*

Proof. (1) Suppose (C, δ) is DG-cotorsion and let $0 \to C^n \to X \to F \to 0$ be an extension in R-Mod with F flat. We consider the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^n & \longrightarrow & X & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C^{n+1} & \longrightarrow & P & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^{n+2} & \longrightarrow & 0 & \longrightarrow & 0 & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Since $\overline{F}[n]$ is a flat complex the preceding sequence of complexes splits, so the sequence of left $R-$ modules $0 \to C^n \to X \to F \to 0$ also splits.

Now $\mathscr{H}\mathrm{om}(F, C)$ is exact if and only if any morphism $f\colon F \to C[n]$ is homotopic to 0. But this is the case because the sequence associated to the mapping cone $0 \to C[n-1] \to M(f) \to F \to 0$ splits (see [6, Lemma 3.2]).

Conversely, suppose C^n is cotorsion for all $n \in \mathbb{Z}$ and $\mathscr{H}\mathrm{om}(F, C)$ is exact for all flat complexes F. Any exact sequence $0 \to C \to P \to F \to 0$ of complexes with F flat splits at the module level. So this sequence is isomorphic to

$$0 \to C \to M(f) \to F \to 0,$$

where $f: F[-1] \to C$ is a morphism. Since $\mathscr{H}om(F[-1], C)$ is exact the sequence $0 \to C \to M(f) \to F \to 0$ splits in $\mathscr{C}$ [6, Lemma 3.2], so

$$0 \to C \to P \to F \to 0$$

also splits.

(2) Follows by [8, 1.12]. ∎

4. EXISTENCE OF FLAT COVERS OF COMPLEXES OVER A COMMUTATIVE NOETHERIAN RING WITH FINITE KRULL DIMENSION

In this section R will denote a commutative noetherian ring with finite Krull dimension d. We are going to follow some ideas of [12, Chaps. 3 and 4] to prove the main result of this section.

PROPOSITION 4.1. *For any flat complex* (F, δ), proj.dim$(F) \leq d$ *in* $\mathscr{C}$.

Proof. By the dual of [5, Theorem 1.5], a complex (C, δ) has proj.dim$(C) \leq n$ in $\mathscr{C}$ if and only if C is exact and proj.dim(Ker$(\delta^k)) \leq n$ in R-Mod for all $k \in \mathbb{Z}$. So we only have to apply this result because a flat complex is exact and Ker(δ^i) is flat, and so by [12, Theorem 4.2.8] proj.dim(Ker$(\delta^i)) \leq d$ for all $i \in \mathbb{Z}$. ∎

LEMMA 4.2. *If R is a ring such that any exact complex has a flat cover, then any complex has a flat cover.*

Proof. Let C be any complex and let $E \to C$ be an exact cover of C (see [6, Theorem 3.18]). We take $F \to E$ to be a flat cover of E. Then, since any flat complex is exact, it is easy to see that $F \to E \to C$ is a flat precover of C. ∎

PROPOSITION 4.3. *Let (C, δ) be a cotorsion complex. Then C has a flat precover with kernel cotorsion.*

Proof. For any $i \in \mathbb{Z}$, we consider the short exact sequence

$$0 \to \operatorname{Ker}(\delta^i) \to C^i \to \operatorname{Ker}(\delta^{i+1}) \to 0.$$

Given $F \to \operatorname{Ker}(\delta^{i+1})$ with F flat, there is a lifting $F \to C^i$ since

$$\operatorname{Ext}^1\big(F, \operatorname{Ker}(\delta^i)\big) = 0.$$

Hence we can construct the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & L \oplus F & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}(\delta^i) & \longrightarrow & C^i & \longrightarrow & \operatorname{Ker}(\delta^{i+1}) & \longrightarrow & 0 \end{array}$$

where $L \to \operatorname{Ker}(\delta^i)$ and $F \to \operatorname{Ker}(\delta^{i+1})$ are flat covers. It is easy to see that the preceding is a flat precover. Also it is clear that the kernel of this precover is cotorsion. Pasting together all the diagrams $\forall\ i \in \mathbb{Z}$, we obtain a flat precover of C whose kernel is cotorsion. ∎

LEMMA 4.4 (see [12, Lemma 3.1.3]). *Let* $0 \to C \to G \to H \to 0$ *be a short exact sequence in* $\mathscr{C}$ *such that* H *is flat and* G *is cotorsion. Then* C *has a flat precover with kernel cotorsion.*

Proof. We consider $F \to G$ a flat precover with kernel cotorsion and the pull-back diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & K & = & K & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & X & \longrightarrow & F & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since F and H are flat, X is flat, also, K is cotorsion, and therefore $X \to C$ is a flat precover with kernel cotorsion. ∎

LEMMA 4.5. *Let* $0 \to B \to C \to D \to 0$ *be a short exact sequence in* $\mathscr{C}$ *with* C *cotorsion and* D *with a flat precover with kernel cotorsion. Then* B *has a flat precover with kernel cotorsion.*

Proof. Let $F \to D$ be a flat precover with kernel cotorsion and consider the pull-back diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since K and C are cotorsion, G is cotorsion. Now we apply Lemma 4.4 to the sequence $0 \to B \to G \to F \to 0$. ∎

THEOREM 4.6. Let R be a commutative noetherian ring with $K-\dim(R) = d$. Then any complex of R-modules has a flat cover.

Proof. By Lemma 4.2 we only need to prove the result for exact complexes. Let C be an exact complex and let

$$0 \to C \to E_0 \to E_1 \to \cdots \to E_d \to D \to 0$$

be an exact sequence with injective complexes E_i. We claim that D is cotorsion. It is clear that D is exact. Now, given $i \in \mathbb{Z}$, we have $0 = H^{-n+i}(C) = \operatorname{Ext}^n_{\mathscr{C}}(\underline{R}[i], C)$ for all $n \geq 1$. Also, for any complex (M, γ), $\operatorname{Hom}_{\mathscr{C}}(\underline{R}[-i], M) \cong \operatorname{Ker}(\gamma^i)$. Hence, if we apply $\operatorname{Hom}_{\mathscr{C}}(\underline{R}[-i], -)$ to the preceding partial injective resolution of C we get

$$0 \to \operatorname{Ker}\left(\delta^i_C\right) \to \operatorname{Ker}\left(\delta^i_{E_0}\right) \to \cdots \to \operatorname{Ker}\left(\delta^i_{E_d}\right) \to \operatorname{Ker}\left(\delta^i_D\right) \to 0$$

an exact sequence with $\operatorname{Ker}(\delta^i_{E_j})$ injective in R-Mod for all $j = 0, \dots, d$. If F is a flat R-module we have by [12, Theorem 4.2.8] that $0 = \operatorname{Ext}^{d+2}(F, \operatorname{Ker}(\delta^i_C)) = \operatorname{Ext}^1(F, \operatorname{Ker}(\delta^i_D))$. So $\operatorname{Ker}(\delta^i_D)$ is cotorsion for all $i \in \mathbb{Z}$.

Note that $E_0, E_1, \dots, E_d, D$ are cotorsion. Now splitting the sequence into short exact sequences and applying Lemma 4.5 repeatedly, we see that C has a flat cover. ∎

Examples. (1) If $0 \to F \to G \to M \to 0$ is an exact sequence of R-modules with F flat and $D \to M$ is a flat cover of M, then the pull-back exact sequence $0 \to F \to X \to D \to 0$ is a flat cover of the original one.

(2) If $0 \to N \to M \to F \to 0$ is an exact sequence of R-modules with F flat and $H \to M$ is a flat cover of M, then the pull-back sequence $0 \to X \to H \to F \to 0$ is a flat cover of the original one.

5. DG-FLAT COVERS

Remember [1] that a complex F is called DG-flat if F^n is flat $\forall n \in \mathbb{Z}$ and for any exact complex E of right R-modules, the complex $E \otimes F$ is exact. A complex D is called DG-injective (resp. DG-projective) if D^n is injective (resp. projective) $\forall\ n \in \mathbb{Z}$ and for any exact complex E, the complex $\mathscr{H}\mathrm{om}(E, D)$ is exact (resp. $\mathscr{H}\mathrm{om}(D, E)$ is exact).

For the next lemma, we recall that the concepts of preenvelope, envelope, and special preenvelope with respect to some class are dual to those of Definition 3.1 (i.e., precover, cover, and special precover respectively).

Lemma 5.1. *Any complex has a special exact preenvelope.*

Proof. Let C be a complex. By [6], we know C has an exact preenvelope $C \to E$ which is a monomorphism because $C \to E(C)$ is a monomorphism and $E(C)$ is exact. Let $0 \to C \to E \to K \to 0$ and consider $0 \to D \to P \to K \to 0$ with P DG-projective and D exact (cf. [11, 2, 6]). We form the pull-back diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & D \arrow[r, equal] \arrow[d] & D \arrow[d] & \\
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & E \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Now the sequence $0 \to C \to G \to P \to 0$ gives a special exact preenvelope. ∎

LEMMA 5.2. *A complex F is DG-flat if and only if*

(i) F^n *is flat for all* $n \in \mathbb{Z}$, *and*

(ii) $\mathscr{H}\mathrm{om}(F, C)$ *is exact for every cotorsion complex* C.

Proof. Suppose F is DG-flat. By Lemma 5.1, we can consider the exact $0 \to F \to E \to P \to 0$ where E is exact and P is DG-projective. Then E is DG-flat and exact. Hence E is flat (E^+ is DG-injective and exact and then by [2, Corollary 6.2.5] or [6, Prop. 3.7], E^+ is injective, so E is flat). Now it is clear that F verifies (i) and (ii). Conversely, suppose F verifies (i) and (ii). Given L an exact complex of right R-modules, then L^+ is cotorsion. Hence $\mathscr{H}\mathrm{om}(F, L^+) \cong (L \otimes F)^+$ is exact, so $L \otimes F$ is exact. ∎

Following [12], we will denote by $\mathscr{F}^\perp$ the class of complexes

$$\left\{C \middle| \mathrm{Ext}^1_{\mathscr{C}}(F, C) = 0 \ \forall\, F \in \mathscr{F}\right\}.$$

LEMMA 5.3. *Let $\mathscr{F}$ be the class of DF-flat complexes and $\mathscr{B}$ the class of cotorsion complexes. Then*

(i) $\mathscr{F}^\perp = \mathscr{B}$.

(ii) $\mathscr{F} = {}^\perp(\mathscr{F}^\perp)$ (*where* ${}^\perp(\mathscr{F}^\perp) = \{D \mid \mathrm{Ext}^1_{\mathscr{C}}(D, C) = 0\ \forall\ C \in \mathscr{F}^\perp\}$.

Proof. (i) It is easy to see that if F is a flat module then $\underline{F}[n]$ is DG-flat. Then, given $(C, \delta) \in \mathscr{F}^\perp$, $H^{-n+1}(C) = \mathrm{Ext}^1_{\mathscr{C}}(\underline{R}[n], C) = 0$, so C is exact. Now we are going to prove that $\mathrm{Ker}(\delta^n)$ is cotorsion for all $n \in \mathbb{Z}$. Given F a flat module, we take $0 \to K \xrightarrow{\alpha} P \to F \to 0$ a projective presentation of F and f: $K \to \mathrm{Ker}(\delta^n)$ a linear map. We can induce a short exact sequence in $\mathscr{C}$, $0 \to \underline{K}[-n] \to^{\underline{\alpha}} \underline{P}[-n] \to \underline{F}[-n] \to 0$, and a morphism $\underline{f}$: $\underline{K}[n] \to C$. Since $\underline{F}[-n]$ is DG-flat, we get a morphism g: $\underline{P}[-n] \to C$ such that $g\underline{\alpha} = \underline{f}$. Then $g^n(P) \subseteq \mathrm{Ker}(\delta^n)$ and $g^n\alpha = f$.

Conversely, let $C \in \mathscr{B}$ and let $0 \to C \to X \to F \to 0$ be an extension with F DG-flat. Since this sequence splits at the module level, the sequence is isomorphic to $0 \to C \to M(h) \to F \to 0$ where h: $F \to C[1]$ is a morphism. By hypothesis, h is homotopic to 0 so the original sequence splits.

(ii) Clearly $\mathscr{F} \subseteq {}^\perp(\mathscr{F}^\perp)$. If $F \in {}^\perp(\mathscr{F}^\perp)$ then $\mathrm{Ext}^1_{\mathscr{C}}(F, C) = 0$ for all $C \in \mathscr{B}$. Hence $\mathscr{H}\mathrm{om}(F, C)$ is exact for all $C \in \mathscr{B}$. Also, given a short exact sequence in $\mathscr{C}$ $0 \to K \to^{\alpha} P \to F \to 0$ with P projective, we take C a cotorsion module and let f: $K^n \to C$ be a linear map. We can induce a morphism $\bar{f}$: $K \to \overline{C}[-n]$. Since $\overline{C}[-n]$ is cotorsion, we get g: $P \to \overline{C}[-n]$ such that $g\alpha = \bar{f}$. So $g^n\alpha^n = f$. ∎

THEOREM 5.4. *Any complex over a commutative noetherian ring with finite Krull dimension has a DG-flat cover which is a quasi-isomorphism.*

Proof. First, by [1], the class of DG-flat complexes is closed under direct limits. So we only need to find DG-flat precovers. Let C be a complex and let $0 \to C \to E \to P \to 0$ be a special exact preenvelope. Also, we consider $0 \to K \to F \to E \to 0$ an exact sequence with F flat and K-cotorsion (since E is exact, by applying the ideas of the proof of Theorem 4.6 we can find such a sequence). We form the pull-back diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & c & \longrightarrow & E & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since F is flat and P is DG-projective, it follows that L is DG-flat. Since K is cotorsion, by Lemma 5.3 we get that $L \to C$ is a DG-flat precover with exact kernel. But then a DG-flat cover is a retract of L and its kernel is a retract of K and so is exact. Hence a DG-flat cover is a quasi-isomorphism. ∎

Remark. The DG-flat cover of an exact complex coincides with the flat cover.

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